A HAMILTONIAN PATH INTEGRAL FOR A DEGENERATE PARABOLIC PSEUDO-DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper, using a Hamiltonian path integral, we give an expression of the symbol of the fundamental solution for a degenerate parabolic pseudo-differential operator. This Hamiltonian path integral converges in the topology of the symbol class $S^{2m}_{\lambda,\rho,\delta}$ and in the weak topology of the symbol class $S^{0}_{\lambda,\rho,\delta}$.

0. Introduction

In this paper, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator in a different way from that in C.Tsutsumi [10]. In [10], she constructed the fundamental solution by Levi-Mizohata method. On the other hand, in this paper, we construct the fundamental solution by a Hamiltonian path integral. If we use a Hamiltonian path integral, we can actually write the symbol of the fundamental solution. Furthermore, this Hamiltonian path integral converges in the topology of the symbol class $S^{2m}_{\lambda,\rho,\delta}$ and in the weak topology of the symbol class $S^{0}_{\lambda,\rho,\delta}$.

In Section 1, we introduce some basic properties of pseudo-differential operators, which we use in Section 2. For the details, see Chapter 7 § 1 and § 2 in H.Kumano-go [6]. In Section 2, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator by a Hamiltonian path integral. Theorem 2.1 is the main theorem in this paper.

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1. Pseudo-Differential Operators

For $x=(x_1,\dots,x_n)\in \boldsymbol{R}^n_x$, $\xi=(\xi_1,\dots,\xi_n)\in \boldsymbol{R}^n_\xi$ and multi-indices of non-negative integers $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta_1,\dots,\beta_n)$, we employ the usual notation:

$$|\alpha|=\alpha_1+\cdots+\alpha_n,\quad |\beta|=\beta_1+\cdots+\beta_n,$$

$$\alpha!=\alpha_1!\cdots\alpha_n!,\quad \beta!=\beta_1!\cdots\beta_n!,$$

$$x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n,\quad \langle x\rangle=(1+|x|^2)^{1/2},\quad \langle\xi\rangle=(1+|\xi|^2)^{1/2},$$

$$\partial_{\xi_j}=\frac{\partial}{\partial\xi_j},\quad D_{x_j}=-i\frac{\partial}{\partial x_j},\quad \partial_\xi^\alpha=\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n},\quad D_x^\beta=D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n}.$$

$\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞-functions on $\boldsymbol{R}^n$. For $u\in S$, we define semi-norms $|u|_{l,\mathcal{S}}(l=0,1,2,\dots)$ by

$$|u|_{l,\mathcal{S}}\equiv\max_{k+|\alpha|\le l}\sup_x|\langle x\rangle^k\partial_x^\alpha u(x)|\quad(l=0,1,2,\dots).$$

Then, $\mathcal{S}$ is a Fréchet space with these semi-norms.
For simplicity, we set $đ\eta\equiv(2\pi)^{-n}d\eta$ and $đ\xi\equiv(2\pi)^{-n}d\xi$.
Oscillatory integral of a function $a(\eta,y)$, is defined by the equality

$$\mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta}a(\eta,y)dy\,đ\eta\equiv\lim_{\epsilon\to0}\iint e^{-iy\cdot\eta}\chi(\epsilon\eta,\epsilon y)a(\eta,y)dy\,đ\eta\,,$$

where $\chi(\eta,y)\in\mathcal{S}$ in $\boldsymbol{R}^{2n}_{\eta,y}$ and $\chi(0,0)=1$. For the details, see Chapter 1 § 6 in H.Kumano-go [6].

Definition 1.1 (A weight function $\lambda(\xi)$).

We say that a real-valued C^∞-function $\lambda(\xi)$ on $\boldsymbol{R}^n_\xi$ is a weight function, if there exist constants $A_0, A_\alpha>0$ such that

$$1\le\lambda(\xi)\le A_0\langle\xi\rangle\,, \tag{1.1}$$

$$|\partial_\xi^\alpha\lambda(\xi)|\le A_\alpha\lambda(\xi)^{1-|\alpha|}\,. \tag{1.2}$$

Examples.

1° $\lambda(\xi) = \langle \xi \rangle$.

2° $\lambda(\xi) = \left\{1 + \sum_{j=1}^{n} |\xi_j|^{2m_j}\right\}^{1/(2m)}$, $(m_j \in \mathbb{N},\ m \equiv \max_{1\le j\le n}\{m_j\})$.

Definition 1.2 (Pseudo-differential operators $S^m_{\lambda,\rho,\delta}$).

We say that a C^∞-function $p(x,\xi)$ on $\boldsymbol{R}^{2n}_{x,\xi}$ is a symbol of class $S^m_{\lambda,\rho,\delta}$ ($m \in \boldsymbol{R}$, $0 \le \delta \le \rho \le 1$), if for any α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\lambda(\xi)^{m+\delta|\beta|-\rho|\alpha|}, \tag{1.3}$$

where $p^{(\alpha)}_{(\beta)}(x,\xi) \equiv \partial^\alpha_\xi D^\beta_x p(x,\xi)$.

The pseudo-differential operator $p(X,D_x)$ with the symbol $p(x,\xi)$ is defined by

$$p(X,D_x)u(x) \equiv \iint e^{i(x-x')\cdot\xi} p(x,\xi)u(x')dx'đ\xi \quad (u \in \mathcal{S}), \tag{1.4}$$

where $đ\xi \equiv (2\pi)^{-n}d\xi$.

Remark.

1° For simplicity, we set $p^{(\alpha)}_{(\beta)}(x,\xi) \equiv \partial^\alpha_\xi D^\beta_x p(x,\xi)$, $p^{(\alpha)}(x,\xi) \equiv \partial^\alpha_\xi p(x,\xi)$ and $p_{(\beta)}(x,\xi) \equiv D^\beta_x p(x,\xi)$ for any α, β.

2° The symbol class $S^m_{\lambda,\rho,\delta}$ is a Fréchet space with the semi-norms

$$|p|^{(m)}_l \equiv \max_{|\alpha+\beta|\le l} \sup_{(x,\xi)}\{|p^{(\alpha)}_{(\beta)}(x,\xi)|\lambda(\xi)^{-(m+\delta|\beta|-\rho|\alpha|)}\} \quad (l = 0,1,2,\dots). \tag{1.5}$$

3° The continuity of $p(X,D_x) : \mathcal{S} \longrightarrow \mathcal{S}$ is clear. Furthermore, we can extend $p(X,D_x) : \mathcal{S} \longrightarrow \mathcal{S}$ to $p(X,D_x) : \mathcal{S}' \longrightarrow \mathcal{S}'$ by means of

$$(p(X,D_x)u, v) \equiv (u, p(X,D_x)^* v) \quad \text{for} \quad u \in \mathcal{S}', v \in \mathcal{S}. \tag{1.6}$$

Theorem 1.3 (Multi-products).

Let M be a positive constant and let $\{m_j\}_{j=1}^{\infty}$ be a sequence of real numbers satisfying

$$\sum_{j=1}^{\infty} |m_j| \le M < \infty . \tag{1.7}$$

For any $\nu = 1, 2, \dots$ and $p_j(x,\xi) \in S^{m_j}_{\lambda,\rho,\delta} (j = 1, 2, \dots, \nu+1)$, there exists $q_{\nu+1}(x,\xi) \in S^{\bar{m}_{\nu+1}}_{\lambda,\rho,\delta} (\bar{m}_{\nu+1} \equiv m_1 + m_2 + \cdots + m_{\nu+1})$ such that

$$q_{\nu+1}(X, D_x) = p_1(X, D_x) p_2(X, D_x) \cdots p_{\nu+1}(X, D_x) . \tag{1.8}$$

Furthermore, for any l, there exist a constant A_l and an integer l' such that

$$|q_{\nu+1}|_l^{(\bar{m}_{\nu+1})} \le (A_l)^{\nu} \prod_{j=1}^{\nu+1} |p_j|_{l'}^{(m_j)} , \tag{1.9}$$

where A_l and l' depend only on M and l, but are independent of ν.

Proof. See Theorem 2.4 in Chapter 7 § 2 of H.Kumano-go [6]. □

Theorem 1.4.

Let $p_j(x,\xi) \in S^{m_j}_{\lambda,\rho,\delta} (j = 1, 2)$. Define $q_\theta(x,\xi)$ $(|\theta| \le 1)$ by

$$q_\theta(x,\xi) \equiv \mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta} p_1(x, \xi + \theta\eta) p_2(x + y, \xi) dy đ\eta . \tag{1.10}$$

Then $\{q_\theta(x,\xi)\}_{|\theta|\le 1}$ is a bounded set of $S^{m_1+m_2}_{\lambda,\rho,\delta}$. Furthermore, for any l, there exist a constant A'_l and an integer l' independent of θ such that

$$|q_\theta|_l^{(m_1+m_2)} \le A'_l |p_1|_{l'}^{(m_1)} |p_2|_{l'}^{(m_2)} . \tag{1.11}$$

Proof. See Lemma 2.4 in Chapter 2 § 2 or Lemma 2.2 in Chapter 7 §2 of H.Kumano-go [6]. □

2. The Main Theorem

Theorem 2.1 (The main theorem).

Let $K(t,x,\xi) \in \mathcal{B}^0([0,T]; S^m_{\lambda,\rho,\delta})$ $(m > 0, 0 \le \delta < \rho \le 1)$. *Assume that* $K(t,x,\xi)$ *satisfies the following conditions* (a1), (a2) *:*

(a1) *There exist constants* $c > 0$ *and* $m'(0 \le m' \le m)$ *such that*

$$Re\ K(t,x,\xi) \le -c\lambda(\xi)^{m'} \text{ on } [0,T] \times \boldsymbol{R}^{2n}_{x,\xi}\,. \tag{2.1}$$

(a2) *For any* α, β, *there exists a constant* $C_{\alpha,\beta}$ *such that*

$$|K^{(\alpha)}_{(\beta)}(t,x,\xi)/Re\ K(t,x,\xi)| \le C_{\alpha,\beta}\lambda(\xi)^{\delta|\beta|-\rho|\alpha|} \text{ on } [0,T] \times \boldsymbol{R}^{2n}_{x,\xi}\,. \tag{2.2}$$

Then we have the following (1) – (5) *:*

(1) *Let* $\varDelta_{t,s} : (T \ge) t \equiv t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1} \equiv s (\ge 0)$ *be an arbitrary division of interval* $[s,t]$ *into subintervals, and let* $e^{(t_j - t_{j+1})K(t_{j+1})}(X, D_x)$ *be an operator defined by*

$$e^{(t_j-t_{j+1})K(t_{j+1})}(X,D_x)u(x) \equiv \iint e^{i(x-x')\cdot\xi} e^{(t_j-t_{j+1})K(t_{j+1},x,\xi)} u(x')dx'\,đ\xi\,. \tag{2.3}$$

Then there exists $p(\varDelta_{t,s}; x, \xi) \in S^0_{\lambda,\rho,\delta}$ *such that*

$$p(\varDelta_{t,s}; X, D_x) = e^{(t-t_1)K(t_1)}(X,D_x)e^{(t_1-t_2)K(t_2)}(X,D_x)\cdots e^{(t_\nu - s)K(s)}(X,D_x)\,. \tag{2.4}$$

(2) *There exist constants* C_l, C'_l *and an integer* l' *such that*

$$|p(\varDelta_{t,s})|^{(0)}_l \le C_l\,, \tag{2.5}$$

and

$$\begin{aligned} &|p(\varDelta_{t,s}) - p(\varDelta'_{t,s})|^{(2m)}_l \\ &\le C'_l(t-s)\Big(|\varDelta_{t,s}| + \sup_{|t'-t''| \le |\varDelta_{t,s}|} |K(t') - K(t'')|^{(m)}_{l'}\Big)\,. \end{aligned} \tag{2.6}$$

Here, $\varDelta_{t,s} : (T \ge) t \equiv t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1} \equiv s(\ge 0)$ *is an arbitrary division of interval* $[s,t]$ *into subintervals,* $\varDelta'_{t,s}$ *is an arbitrary refinement of* $\varDelta_{t,s}$,

$|\Delta_{t,s}|$ *denotes the size of division defined by* $|\Delta_{t,s}| \equiv \max_{0\le j\le \nu} |t_j - t_{j+1}|$, *and the constants* C_l, C_l' *and the integer* l' *are independent of* ν, $\Delta_{t,s}$ *and* $\Delta'_{t,s}$.

(3) *There exists* $p^\star(t,s;x,\xi) \in S^0_{\lambda,\rho,\delta}$ *such that* $p(\Delta_{t,s};x,\xi)$ $(\in S^0_{\lambda,\rho,\delta})$ *converges to* $p^\star(t,s;x,\xi)$ $(\in S^0_{\lambda,\rho,\delta})$ *in* $S^{2m}_{\lambda,\rho,\delta}$ *as* $|\Delta_{t,s}|$ *tends to* 0.

Furthermore, $p^\star(t,s;x,\xi)$ *has the following expression:*

$$p^\star(t,s;x,\xi) = \lim_{|\Delta_{t,s}|\to 0} \mathrm{O_s}\text{—}\iint\cdots\iint e^{-i\sum_{j=1}^{\nu} y^j\cdot\eta^j}$$

$$\times \exp\left(\sum_{j=0}^{\nu}(t_j - t_{j+1})K(t_{j+1}, x+\bar{y}^j, \xi+\eta^{j+1})\right)dy^1 đ\eta^1\cdots dy^\nu đ\eta^\nu\,, \tag{2.7}$$

where $\bar{y}^0 \equiv 0$, $\bar{y}^j \equiv y^1 + y^2 + \cdots + y^j$, *and* $\eta^{\nu+1} \equiv 0$.

(4) *For* $u \in L^2$, *the pseudo-differential operator* $U(t,s) \equiv p^\star(t,s;X,D_x)$ *satisfies the following relation:*

$$\begin{aligned}
&U(t,s)u(x)\\
&= \lim_{|\Delta_{t,s}|\to 0} e^{(t-t_1)K(t_1)}(X,D_x)e^{(t_1-t_2)K(t_2)}(X,D_x)\cdots e^{(t_\nu - s)K(s)}(X,D_x)u(x)\\
&= \lim_{|\Delta_{t,s}|\to 0}\iint\cdots\iint \exp\left(\sum_{j=0}^{\nu} i(x^j - x^{j+1})\cdot\xi^{j+1} + (t_j - t_{j+1})K(t_{j+1}, x^j, \xi^{j+1})\right)\\
&\qquad\qquad \times u(x^{\nu+1})dx^{\nu+1}đ\xi^{\nu+1}\cdots dx^1 đ\xi^1\,,
\end{aligned} \tag{2.8}$$

in L^2 *where* $x^0 \equiv x$.

(5) $U(t,s) \equiv p^\star(t,s;X,D_x)$ *is the fundamental solution for the operator* $L \equiv \partial_t - K(t,X,D_x)$ *such that*

$$\begin{cases} LU(t,s) = 0 & \text{on } [s,T] \\ U(s,s) = I & (0 \le s \le T). \end{cases} \tag{2.9}$$

Remark.

1° It is sufficient to satisfy the conditions $(a1)$ and $(a2)$ for $|\xi| \ge M$, with a constant $M \ge 0$. In fact, in this case, there exists a sufficiently large $R > 0$ such that the symbol $K_R(t,x,\xi) \equiv K(t,x,\xi) - R$ satisfies $(a1)$ and $(a2)$ for any ξ. Let $U_R(t,s)$ be the fundamental solution of $L_R \equiv \partial_t - K_R(t,X,D_x)$. Then $U(t,s) \equiv e^{(t-s)R}U_R(t,s)$ is the fundamental solution of L.

2° We can replace $(t_j - t_{j+1})K(t_{j+1},\cdot,\cdot)$ with $\int_{t_{j+1}}^{t_j} K(\tau,\cdot,\cdot)d\tau$. Furthermore, in this case, we can replace (2.6) with

$$|p(\Delta_{t,s}) - p(\Delta'_{t,s})|_l^{(2m)} \le C_l'(t-s)|\Delta_{t,s}|\,, \tag{2.6'}$$

and the proof of Theorem 2.1 becomes a little easier.

Example.

Consider

$$L \equiv \partial_t + a(t)|x|^{2l}(-\,\triangle)^m + (-\,\triangle)^{m'} \quad (0 \le a(t) \in \mathcal{C}[0,T]\,,\, m - m' < l)\,.$$

If we set $\rho = 1$, $\delta = (m - m')/l$, $m \to 2m$ and $m' \to 2m'$, then the symbol $a(t)|x|^{2l}|\xi|^{2m} + |\xi|^{2m'}$ satisfies the conditions $(a1)$ and $(a2)$. Therefore, we see that these conditions are satisfied not only by the usual parabolic operators, but also by parabolic operators of a degenerate type.

Before we prove Theorem 2.1, we prepare some lemmas:
To begin with, for $T \ge t \ge s \ge 0$, we define $p(t,s;x,\xi)$ by

$$p(t,s;x,\xi) \equiv \exp\Big((t-s)K(s,x,\xi)\Big)\,. \tag{2.10}$$

The next lemma is a generalization of asymptotic expansion formulas, and an essential part in this paper. Especially, it is important that all constants are independent of $\varDelta_{t_0,t_{\nu+1}}$ and ν.

Lemma 2.2 (Key Lemma).

Let $\varDelta_{t_0,t_{\nu+1}} : (T \ge)t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1}(\ge 0)$, $\nu = 1,2,\ldots$, and let N_0 be a fixed positive integer such that $(\rho-\delta)N_0 \ge 2m$. Define $q(\varDelta_{t_0,t_1};x,\xi)$, $q(\varDelta_{t_0,t_{\nu+1}};x,\xi)$, and $r(\varDelta_{t_0,t_{\nu+1}};x,\xi)$ respectively by

$$q(\varDelta_{t_0,t_1};x,\xi) \equiv p(t_0,t_1;x,\xi)\,, \tag{2.11}$$

$$\begin{aligned} q(\varDelta_{t_0,t_{\nu+1}};x,\xi) \equiv & \sum_{|\alpha^1|+|\alpha^2|+\cdots+|\alpha^\nu|<N_0} \frac{1}{\alpha^1!\alpha^2!\cdots\alpha^\nu!} \\ & \times p_{(\alpha^\nu)}(t_\nu,t_{\nu+1};x,\xi)\partial_\xi^{\alpha^\nu}\Big(p_{(\alpha_{\nu-1})}(t_{\nu-1},t_\nu;x,\xi)\partial_\xi^{\alpha^{\nu-1}}\Big(\\ & \cdots p_{(\alpha^2)}(t_2,t_3;x,\xi)\partial_\xi^{\alpha^2}\Big(p_{(\alpha^1)}(t_1,t_2;x,\xi)\partial_\xi^{\alpha^1}\Big(p(t_0,t_1;x,\xi)\Big)\Big)\cdots\Big)\Big)\,. \end{aligned} \tag{2.12}$$

and

$$r(\Delta_{t_0,t_{\nu+1}};x,\xi) \equiv \sum_{|\alpha^1|+|\alpha^2|+\cdots+|\alpha^\nu|=N_0,\,|\alpha^\nu|\neq 0} \frac{|\alpha^\nu|}{\alpha^1!\alpha^2!\cdots\alpha^\nu!}$$
$$\times \int_0^1 (1-\theta)^{|\alpha^\nu|-1} \mathrm{O}_s\text{—}\iint e^{-iy\cdot\eta} p_{(\alpha^\nu)}(t_\nu,t_{\nu+1};x+y,\xi)$$
$$\times \partial_\xi^{\alpha^\nu}\Big(p_{(\alpha_{\nu-1})}(t_{\nu-1},t_\nu;x,\xi+\theta\eta)\partial_\xi^{\alpha^{\nu-1}}\Big(\cdots p_{(\alpha^2)}(t_2,t_3;x,\xi+\theta\eta)$$
$$\times \partial_\xi^{\alpha^2}\Big(p_{(\alpha^1)}(t_1,t_2;x,\xi+\theta\eta)\partial_\xi^{\alpha^1}\Big(p(t_0,t_1;x,\xi+\theta\eta)\Big)\Big)\cdots\Big)\Big)dy\bar{d}\eta d\theta\,. \tag{2.13}$$

Then it follows that

$$q(\Delta_{t_0,t_\nu};X,D_x)p(t_\nu,t_{\nu+1};X,D_x)$$
$$= q(\Delta_{t_0,t_{\nu+1}};X,D_x) + r(\Delta_{t_0,t_{\nu+1}};X,D_x)\,. \tag{2.14}$$

Furthermore, there exist constants $C_{1,l}, C_{2,l}, C_{3,l}$ *such that*

$$|q(\Delta_{t_0,t_\nu})|_l^{(0)} \le C_{1,l}\,, \tag{2.15}$$

$$|q(\Delta_{t_0,t_{\nu+1}}) - p(t_0,t_{\nu+1})|_l^{(2m)}$$
$$\le C_{2,l}(t_0-t_{\nu+1})\Big((t_0-t_{\nu+1}) + \sup_{t_0\ge t'\ge t''\ge t_{\nu+1}} |K(t')-K(t'')|_l^{(m)}\Big)\,, \tag{2.16}$$

and

$$|r(\Delta_{t_0,t_{\nu+1}})|_l^{(0)} \le C_{3,l}(t_0-t_\nu)(t_\nu-t_{\nu+1})\,, \tag{2.17}$$

for any $\Delta_{t_0,t_{\nu+1}} : (T\ge)t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1}(\ge 0)$ *and* $\nu = 1,2,\ldots$.

Proof.

1° For $T \ge t \ge s \ge 0$, we set

$$\eta(t,s;x,\xi) \equiv -(t-s)\mathrm{Re}\ K(s,x,\xi)\,(\ge 0)\,. \tag{2.18}$$

Furthermore, for $\Delta_{t_0,t_{\nu+1}} : (T\ge)t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1}(\ge 0)$ and $\nu = 1,2,\ldots$, we define $d(\Delta_{t_0,t_\nu};x,\xi)$ by

$$d(\Delta_{t_0,t_\nu};x,\xi) \equiv \prod_{j=0}^{\nu-1} p(t_j,t_{j+1};x,\xi)\,, \tag{2.19}$$

and we set

$$\eta(\Delta_{t_0,t_\nu};x,\xi) \equiv \sum_{j=0}^{\nu-1} \eta(t_j,t_{j+1};x,\xi)\,. \tag{2.20}$$

Clearly, we have

$$|d(\Delta_{t_0,t_\nu};x,\xi)| = \exp\Big(-\eta(\Delta_{t_0,t_\nu};x,\xi)\Big)\,. \tag{2.21}$$

2° Define $d_{\alpha,\beta}(\Delta_{t_0,t_\nu};x,\xi)$ by

$$d^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_\nu};x,\xi) \equiv d_{\alpha,\beta}(\Delta_{t_0,t_\nu};x,\xi)d(\Delta_{t_0,t_\nu};x,\xi)\,. \tag{2.22}$$

Then, by induction, for any α,β ($|\alpha+\beta|\geq 1$) and α',β', there exists a constant $C_{\alpha,\beta,\alpha',\beta'}$ such that

$$\begin{aligned}|d_{\alpha,\beta}{}^{(\alpha')}_{(\beta')}(\Delta_{t_0,t_\nu};x,\xi)| \leq C_{\alpha,\beta,\alpha',\beta'}\eta(\Delta_{t_0,t_\nu};x,\xi)\Big(\eta(\Delta_{t_0,t_\nu};x,\xi)+1\Big)^{|\alpha+\beta|-1} \\ \times\lambda(\xi)^{\delta|\beta+\beta'|-\rho|\alpha+\alpha'|}\,,\end{aligned} \tag{2.23}$$

for any $\Delta_{t_0,t_{\nu+1}} : (T\geq)t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1}(\geq 0)$ and $\nu = 1,2,\ldots$.

3° Let $\widetilde{\alpha}^\nu \equiv (\alpha^1,\ldots,\alpha^\nu)$ denote a multi-index of $\boldsymbol{R}^{\nu n}$. Define $f_{\widetilde{\alpha}^\nu}(\Delta_{t_0,t_{\nu+1}};x,\xi)$ by

$$\begin{aligned}&f_{\widetilde{\alpha}^\nu}(\Delta_{t_0,t_{\nu+1}};x,\xi)d(\Delta_{t_0,t_{\nu+1}};x,\xi) \\ &\equiv p_{(\alpha^\nu)}(t_\nu,t_{\nu+1};x,\xi)\partial_\xi^{\alpha^\nu}\Big(p_{(\alpha_{\nu-1})}(t_{\nu-1},t_\nu;x,\xi)\partial_\xi^{\alpha^{\nu-1}}\Big(\\ &\cdots p_{(\alpha^2)}(t_2,t_3;x,\xi)\partial_\xi^{\alpha^2}\Big(p_{(\alpha^1)}(t_1,t_2;x,\xi)\partial_\xi^{\alpha^1}\Big(p(t_0,t_1;x,\xi)\Big)\Big)\cdots\Big)\Big)\,.\end{aligned} \tag{2.24}$$

Then, by induction, for any $N = 1,2,\ldots$ and α,β, there exists a constant $C_{N,\alpha,\beta}$ such that

$$\begin{aligned}|f_{\widetilde{\alpha}^\nu}{}^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_{\nu+1}};x,\xi)| \leq C_{N,\alpha,\beta}\Big(\prod_{k=1}^{J}\eta(t_{j_k},t_{j_k+1};x,\xi)\Big)\eta(\Delta_{t_0,t_{\nu+1}};x,\xi) \\ \times\Big(\eta(\Delta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{2(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|}\,,\end{aligned} \tag{2.25}$$

where

$$1\leq j_1 < j_2 < \cdots < j_J \leq \nu\,,\quad |\alpha^{j_k}| \neq 0\,(k=1,2,\ldots,J)\,,$$

and

$$\sum_{j=1}^{\nu}|\alpha^j| = \sum_{k=1}^{J}|\alpha^{j_k}| = N\,,$$

for any $\varDelta_{t_0,t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1,2,\ldots$.

4° For $N = 1,2,\ldots$, define $g_N(\varDelta_{t_0,t_{\nu+1}}; x,\xi)$ by

$$g_N(\varDelta_{t_0,t_{\nu+1}}; x,\xi) \equiv \sum_{|\alpha^1|+|\alpha^2|+\cdots+|\alpha^\nu|=N} \frac{1}{\alpha^1!\alpha^2!\cdots\alpha^\nu!} f_{\widetilde{\alpha}^\nu}(\varDelta_{t_0,t_{\nu+1}}; x,\xi)\,. \tag{2.26}$$

By (2.25), we have

$$\begin{aligned}
&|g_{N}{}^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_{\nu+1}}; x,\xi)| \\
&\leq \sum_{J=1}^{N} \sum_{1\leq j_1<j_2<\cdots<j_J\leq\nu} \sum_{\sum_{k=1}^{J}|\alpha^{j_k}|=N,\,|\alpha^{j_k}|\neq 0} \frac{1}{\alpha^{j_1}!\alpha^{j_2}!\cdots\alpha^{j_J}!} \\
&\quad\times C_{N,\alpha,\beta}\Big(\prod_{k=1}^{J}\eta(t_{j_k},t_{j_k+1};x,\xi)\Big)\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi) \\
&\quad\times\Big(\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{2(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\
&\leq (nN)^N C_{N,\alpha,\beta}\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)\Big(\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{2(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\
&\quad\times\left(\sum_{J=1}^{N}\sum_{1\leq j_1<j_2<\cdots<j_J\leq\nu}\prod_{k=1}^{J}\eta(t_{j_k},t_{j_k+1};x,\xi)\right).
\end{aligned} \tag{2.27}$$

Hence, for any $N = 1,2,\ldots$ and α,β, there exists a constant $C'_{N,\alpha,\beta}$ such that

$$\begin{aligned}
|g_{N}{}^{(\alpha)}_{(\beta)}(\varDelta_{t_0,t_{\nu+1}}; x,\xi)| &\leq C'_{N,\alpha,\beta}\Big(\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)\Big)^2 \\
&\quad\times\Big(\eta(\varDelta_{t_0,t_{\nu+1}};x,\xi)+1\Big)^{3(N-1)}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|},
\end{aligned} \tag{2.28}$$

for any $\varDelta_{t_0,t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1,2,\ldots$.

5° Set

$$h_N(\varDelta_{t_0,t_{\nu+1}}; x,\xi) \equiv g_N(\varDelta_{t_0,t_{\nu+1}}; x,\xi)\, d(\varDelta_{t_0,t_{\nu+1}}; x,\xi)\,. \tag{2.29}$$

Here we note that

$$\sup_{\eta>0}\eta^k e^{-\eta} < \infty \quad (k = 0,1,2,\ldots)\,. \tag{2.30}$$

By (2.21), (2.23) and (2.28), there exist constants $C'_{\alpha,\beta}$, $C''_{\alpha,\beta}$, $C''_{N,\alpha,\beta}$, $C'''_{N,\alpha,\beta}$, $C''''_{N,\alpha,\beta}$ such that

$$|d^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_\nu};x,\xi)| \le \begin{cases} C'_{\alpha,\beta}\lambda(\xi)^{\delta|\beta|-\rho|\alpha|} \\ C''_{\alpha,\beta}(t_0-t_\nu)\lambda(\xi)^{m+\delta|\beta|-\rho|\alpha|} \quad (|\alpha+\beta|\ge 1), \end{cases} \tag{2.31}$$

and

$$|h_N{}^{(\alpha)}_{(\beta)}(\Delta_{t_0,t_{\nu+1}};x,\xi)| \le \begin{cases} C''_{N,\alpha,\beta}\lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\ C'''_{N,\alpha,\beta}(t_0-t_{\nu+1})\lambda(\xi)^{m-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\ C''''_{N,\alpha,\beta}(t_0-t_{\nu+1})^2\lambda(\xi)^{2m-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|}, \end{cases} \tag{2.32}$$

for any $\Delta_{t_0,t_{\nu+1}} : (T\ge)t_0 \ge t_1 \ge \cdots \ge t_\nu \ge t_{\nu+1}(\ge 0)$ and $\nu = 1,2,\ldots$.

6° Now we note that

$$q(\Delta_{t_0,t_{\nu+1}};x,\xi) = d(\Delta_{t_0,t_{\nu+1}};x,\xi) + \sum_{N=1}^{N_0-1} h_N(\Delta_{t_0,t_{\nu+1}};x,\xi), \tag{2.33}$$

and

$$\begin{aligned} &d(\Delta_{t_0,t_{\nu+1}};x,\xi) - p(t_0,t_{\nu+1};x,\xi) \\ &= \sum_{j=0}^{\nu}(t_j-t_{j+1})\Big(K(t_{j+1},x,\xi)-K(t_{\nu+1},x,\xi)\Big) \\ &\times \int_0^1 \exp\Big(\theta\sum_{j=0}^{\nu}(t_j-t_{j+1})K(t_{j+1},x,\xi)\Big)\exp\Big((1-\theta)(t_0-t_{\nu+1})K(t_{\nu+1},x,\xi)\Big)d\theta. \end{aligned} \tag{2.34}$$

By (2.31) and (2.32), we get (2.15) and (2.16). Furthermore, we note that

$$\begin{aligned} r(\Delta_{t_0,t_{\nu+2}};x,\xi) = &\sum_{0<|\alpha^{\nu+1}|<N_0} \frac{|\alpha^{\nu+1}|}{\alpha^{\nu+1}!}\int_0^1 (1-\theta)^{|\alpha^{\nu+1}|-1} \\ &\times \mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta} h_{N_0-|\alpha^{\nu+1}|}{}^{(\alpha^{\nu+1})}(\Delta_{t_0,t_{\nu+1}};x,\xi+\theta\eta) \\ &\times d_{(\alpha^{\nu+1})}(\Delta_{t_{\nu+1},t_{\nu+2}};x+y,\xi)dy\,đ\eta\, d\theta \\ &+ \sum_{|\alpha^{\nu+1}|=N_0} \frac{|\alpha^{\nu+1}|}{\alpha^{\nu+1}!}\int_0^1 (1-\theta)^{|\alpha^{\nu+1}|-1} \\ &\times \mathrm{O_s}\text{—}\iint e^{-iy\cdot\eta} d^{(\alpha^{\nu+1})}(\Delta_{t_0,t_{\nu+1}};x,\xi+\theta\eta) \\ &\times d_{(\alpha^{\nu+1})}(\Delta_{t_{\nu+1},t_{\nu+2}};x+y,\xi)dy\,đ\eta\, d\theta. \end{aligned} \tag{2.35}$$

By (2.31), (2.32) and Theorem 1.4 , we get (2.17).

7° By induction, we get (2.14). □

The idea of the next lemma is found in Fujiwara [3].

Lemma 2.3 (Fujiwara's Skip).

Define $\Upsilon(\Delta_{t_0,t_{\nu+1}};x,\xi)\in S^0_{\lambda,\rho,\delta}$ *by*

$$\begin{aligned}&p(t_0,t_1;X,D_x)p(t_1,t_2;X,D_x)\cdots p(t_\nu,t_{\nu+1};X,D_x)\\&\equiv q(\Delta_{t_0,t_{\nu+1}};X,D_x)+\Upsilon(\Delta_{t_0,t_{\nu+1}};X,D_x).\end{aligned}\tag{2.36}$$

Then it follows that

$$\begin{aligned}\Upsilon(\Delta_{t_0,t_{\nu+1}};X,D_x)=&\sum{}' r(\Delta_{t_0,t_{j_1+1}};X,D_x)r(\Delta_{t_{j_1+1},t_{j_2+1}};X,D_x)\\&\cdots r(\Delta_{t_{j_{J-1}+1},t_{j_J+1}};X,D_x)q(\Delta_{t_{j_J+1},t_{\nu+1}};X,D_x),\end{aligned}\tag{2.37}$$

where $\sum{}'$ *stands for the summation with respect to the sequences of integers* $(j_1,j_2,\ldots,j_J)$ *with the property*

$$0<j_1<j_1+1<j_2<j_2+1<\cdots<j_{J-1}<j_{J-1}+1<j_J\le\nu,\tag{2.38}$$

and, in the special case of $j_J=\nu$*, we set* $q(\Delta_{t_{j_J+1},t_{\nu+1}};X,D_x)\equiv I$*.*
Furthermore, there exists a constant $C_{4,l}$ *such that*

$$|\Upsilon(\Delta_{t_0,t_{\nu+1}})|_l^{(0)}\le C_{4,l}(t_0-t_{\nu+1})^2,\tag{2.39}$$

for any $\Delta_{t_0,t_{\nu+1}}:(T\ge)t_0\ge t_1\ge\cdots\ge t_\nu\ge t_{\nu+1}(\ge0)$ *and* $\nu=1,2,\ldots.$

Proof. Using (2.14) inductively, we get (2.37). Now let A_l,l' be the same constants in Theorem 1.3, and let $C_{1,l},C_{3,l}$ be the same constants in Lemma 2.2. By (2.15), (2.17) and Theorem 1.3, we have

$$\begin{aligned}|\Upsilon(\Delta_{t_0,t_{\nu+1}})|_l^{(0)}&\le\sum{}'(A_l)^J|r(\Delta_{t_0,t_{j_1+1}})|_{l'}^{(0)}|r(\Delta_{t_{j_1+1},t_{j_2+1}})|_{l'}^{(0)}\\&\quad\cdots|r(\Delta_{t_{j_{J-1}+1},t_{j_J+1}})|_{l'}^{(0)}|q(\Delta_{t_{j_J+1},t_{\nu+1}})|_{l'}^{(0)}\\&\le\sum{}'(A_l)^J\Big(\prod_{k=1}^{J}C_{3,l'}(t_0-t_{\nu+1})(t_{j_k}-t_{j_k+1})\Big)C_{1,l'}\\&\le C_{1,l'}\bigg(\prod_{j=0}^{\nu}\Big(1+A_lC_{3,l'}(t_0-t_{\nu+1})(t_j-t_{j+1})\Big)-1\bigg)\\&\le C_{4,l}(t_0-t_{\nu+1})^2.\quad\square\end{aligned}\tag{2.40}$$

Now we prove Theorem 2.1:

Proof of Theorem2.1.

1° Define $p(\Delta_{t,s};x,\xi)$ by

$$p(\Delta_{t,s};x,\xi) \equiv q(\Delta_{t,s};x,\xi) + \Upsilon(\Delta_{t,s};x,\xi) . \tag{2.41}$$

Then (1) is clear.

2° By (2.14) and (2.39), we get (2.5). Next, we note that

$$\begin{aligned} &p(\Delta'_{t_j,t_{j+1}};x,\xi) - p(t_j,t_{j+1};x,\xi) \\ &= \Big(q(\Delta'_{t_j,t_{j+1}};x,\xi) - p(t_j,t_{j+1};x,\xi)\Big) + \Upsilon(\Delta'_{t_j,t_{j+1}};x,\xi) , \end{aligned} \tag{2.42}$$

Hence, by (2.16) and (2.39), there exists a constant $C_{5,l}$ such that

$$\begin{aligned} &|p(t_j,t_{j+1}) - p(\Delta'_{t_j,t_{j+1}})|_l^{(2m)} \\ &\le C_{5,l}(t_j - t_{j+1})\Big((t_j - t_{j+1}) + \sup_{t_j \ge t' \ge t'' \ge t_{j+1}} |K(t') - K(t'')|_l^{(m)}\Big) . \end{aligned} \tag{2.43}$$

Here we can write

$$\begin{aligned} &p(\Delta_{t,s};X,D_x) - p(\Delta'_{t,s};X,D_x) = \sum_{j=0}^{\nu} p(\Delta'_{t_0,t_j};X,D_x) \\ &\circ \Big(p(t_j,t_{j+1};X,D_x) - p(\Delta'_{t_j,t_{j+1}};X,D_x)\Big) \circ p(\Delta_{t_{j+1},t_{\nu+1}};X,D_x) . \end{aligned} \tag{2.44}$$

By (2.5), (2.43) and Theorem 1.3, we get (2.6).

3° By (2.6) and (2.5), there exists $p^\star(t,s;x,\xi) \in S^0_{\lambda,\rho,\delta}$ such that

$$|p^\star(t,s)|_l^{(0)} \le C_l , \tag{2.45}$$

and

$$\begin{aligned} &|p(\Delta_{t,s}) - p^\star(t,s)|_l^{(2m)} \\ &\le C'_l(t-s)\Big(|\Delta_{t,s}| + \sup_{|t'-t''| \le |\Delta_{t,s}|} |K(t') - K(t'')|_{l'}^{(m)}\Big) . \end{aligned} \tag{2.46}$$

Hence we get (3).

4° By the result of (3), we get (4). See Chapter 3 § 7 in H.Kumano-go [6].

5° Using the results of (2) and (3), it is easy to check (5). □

References

[1] R.P.Feynman, *Space-time approach to non relativistic quantum mechanics*, Rev. of Modern Phys. **20** (1948), 367–387.

[2] D.Fujiwara, *A remark on Taniguchi-Kumano-go theorem for product of Fourier integral operators*, Pseudo-differential operators, Proc. Oberwolfach 1986, Lecture notes in Math, Springer **1256** (1987), 135-153.

[3] ———, *The stationary phase method with an estimate of the remainder term on a space of large dimension*, Nagoya Math. J. **124** (1991), 61–97.

[4] ———, *Some Feynman Path Integrals As Oscillatory Integrals Over A Sobolev Manifold*, preprint (1993).

[5] H.Kitada and H.Kumano-go, *A family of Fourier integral operators and the fundamental solution for a Schrödinger equation*, Osaka J. Math. **18** (1981), 291–360.

[6] H.Kumano-go, *Pseudo-Differential Operator*, MIT press, Cambridge, Massachusetts and London, England, 1983.

[7] H.Kumano-go and K.Taniguchi, *Fourier integral operators of multi-phase and the fundamental solution for a hyperbolic system*, Funkcial. Ekvac. **22** (1979), 161–196.

[8] K.Shinkai, *The symbol calculus for the fundamental solution of a degenerate parabolic system with applications*, Osaka J. Math. **14** (1977), 55-84.

[9] K.Taniguchi, *Multi-products of Fourier integral operators and the fundamental solution for a hyperbolic system with involutive characteristics*, Osaka J. Math. **21** (1984), 169–224.

[10] C.Tsutsumi, *The fundamental solution for a degenerate parabolic pseudo-differential operator*, Proc. Japan Acad. **50** (1974), 11–15.

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